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Eigenvalue singular problem of factorized fourth-order self-adjoint differential equations

Dumitru I. Caruntu

University of Texas-Pan American, Mechanical Engineering Department, Edinburg, TX 78539, USA

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ABSTRACT

This paper deals with the eigenvalue singular problem of the spectral type differential equations of the fourth-order self-adjoint differential operators

$$\frac{1}{(1-x)^p(1+x)^q}\frac{d^2}{dx^2}\left[(1-x)^{p+2}(1+x)^{q+2}\frac{d^2y}{dx^2}\right]-\mu y=0,\quad y(-1),\ y(1)\quad \textit{finite}$$

where, $p \geqslant 1,\ q \geqslant 1$. It has been reported in the literature that spectral type differential equations above have general solutions in terms of hypergeometric functions. In this work it is showed that the general solution in terms of hypergeometric functions reduces to Jacobi orthogonal polynomials (as eigenfunctions) in the case of eigenvalue singular problem. The corresponding eigenvalues are found. As application, the natural frequencies and mode shapes of mechanical transverse vibrations of a nonuniform singular structure are reported.

1. Introduction

The goal of this paper is to find the eigenfunctions and eigenvalues of the eigenvalue singular problem of the fourth-order self-adjoint differential equation

$$\frac{1}{(1-x)^p(1+x)^q}\frac{d^2}{dx^2}\left[(1-x)^{p+2}(1+x)^{q+2}\frac{d^2y}{dx^2}\right]-\mu y=0,\quad y(-1),\ y(1)\quad \textit{finite},$$

where $p \geqslant 1,\ q \geqslant 1$, and μ is a positive constant.

Factorizations of fourth-order differential equations have been reported in the literature as follows: (1) the factorization of fourth-order Bessel-type differential equation into a pair of second-order differential operators [1]; (2) the factorization of fourth-order differential equations satisfied by the Laguerre–Hahn orthogonal polynomials obtained from some perturbations of classical orthogonal polynomials [2] (four linearly independent solutions of the fourth-order differential equations have been found and extended from integers to reals for the associated classical orthogonal polynomials with integer order of association); (3) the fourth-order differential equation satisfied by associated polynomials belonging to Hahn–Laguerre class of classical orthogonal polynomials [3] has been written as a sum of two differential operators, of which one fourth-order that has been factorized [4]; (4) the factorization of fourth-order and sixth-order differential equations which corresponded to truncations of higher-order derivatives for Schrődinger equation [5]; (5) the factorization of Euler–Bernoulli fourth-order differential operator describing transverse vibrations of nonuniform beams into a pair of commuting Sturm–Liouville second-order differential operators [6,7]; (6) factorization of Euler–Bernoulli operator provided the coefficients

E-mail addresses: caruntud@utpa.edu, caruntud2@asme.org, dcaruntu@yahoo.com

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of the factors satisfied a system of nonlinear ordinary differential equations [8] (the system reduced to a single nonlinear third-order differential equation called principal equation and analyzed using Lie group methods); (7) the factorization of the fourth-order differential equation of bending vibrations of a class of variable cross-section rotating beams [9] (a technique of computing natural frequencies and mode shapes as functions of setting angle and rotation rate has been reported). Closed form solutions involving classical Jacobi orthogonal polynomials have been reported in the literature for transverse vibrations [10,11], and self-adjoint differential equations [12].

Two very important results related to this paper are as follows. First, classical orthogonal polynomials satisfy even order self-adjoint spectral type differential equations [12]. The factorization method has not been used in [12]. Second, the same even order self-adjoint spectral type differential equation can be factorized [13]. In particular, it has been reported that the above given spectral type differential equation has a general solution in terms of hypergeometric functions.

The contribution of this paper consists of showing that (1) the general solution in terms of hypergeometric functions of the fourth-order self-adjoint differential equation of this work reduces to Jacobi orthogonal polynomials in the case of the eigenvalue singular problem associated to the equation, (2) Refs. [12,13] are in perfect agreement, (3) in applications such as mechanical vibrations of nonuniform beams of parabolic variation and both ends sharp, the boundary value problem results in an eigenvalue singular problem and the mode shape results in Jacobi polynomials. This paper can be very useful as reference to researchers interested in eigenvalue singular problems.

2. Factorization of fourth-order differential equation [13]

Caruntu [13] showed that the fourth-order spectral type differential equation

$$\frac{1}{(1-x)^p(1+x)^q}\frac{d^2}{dx^2}\left[(1-x)^{p+2}(1+x)^{q+2}\frac{d^2y}{dx^2}\right]-\mu y=0, \tag{1}$$

or in its extended form

$$\begin{aligned}&(1-x^2)^2\frac{d^4y}{dx^4}+2(1-x^2)[-(p+q+4)x+q-p]\frac{d^3y}{dx^3}\\&\quad+\left[(p+q+3)(p+q+4)x^2+2(p-q)(p+q+3)x+(q-p)^2-p-q-4\right]\frac{d^2y}{dx^2}-\mu y\\&\quad=0,\end{aligned} \tag{2}$$

where μ is real and positive:

(1) Can be factorized as

$$\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-\lambda_1\right]\cdot\left[\beta\frac{d^2}{dx^2}+(\alpha+\beta')\frac{d}{dx}-\lambda_2\right]y(x)=0, \tag{3}$$

where α and β and are polynomial functions given by

$$\beta(x)=1-x^2,\quad \alpha(x)=-(p+q)x+q-p,\quad p\geqslant 1\quad\text{and}\quad q\geqslant 1, \tag{4}$$

and the constants λ_1, λ_2, and δ_2 satisfy the following equations

$$\begin{cases}\lambda_1+\lambda_2=\delta_2\\ \lambda_1\lambda_2=-\mu\end{cases}, \tag{5}$$

$$\delta_2=\frac{d\alpha}{dx}+\frac{d^2\beta}{dx^2}. \tag{6}$$

Using Eqs. (4)–(6) these constants can be written as

$$\lambda_1=\frac{\delta_2+\sqrt{\delta_2^2+4\mu}}{2},\quad \lambda_2=\frac{\delta_2-\sqrt{\delta_2^2+4\mu}}{2}, \tag{7}$$

$$\delta_2=-p-q-2, \tag{8}$$

(2) Its general solution is found in terms of hypergeometric functions as follows

$$y(x)=A_1\cdot{}_2F_1\left(a_1,b_1,c;\frac{1-x}{2}\right)+A_2\cdot{}_2F_1\left(a_2,b_2,c;\frac{1-x}{2}\right)+B_1\cdot w\left(a_1,b_1,c;\frac{1-x}{2}\right)+B_2\cdot w\left(a_2,b_2,c;\frac{1-x}{2}\right), \tag{9}$$

where A_1,A_2,B_1,B_2 are constants of integration, and a_i,b_i,c are the constant parameters of the canonical form of Gauss equations. Canonical Gauss equations [14] are given by

$$z(1-z)\frac{d^2y}{dz^2}+[c-(a_i+b_i+1)z]\frac{dy}{dz}-a_ib_iz=0. \tag{10}$$

A variable change $x=1-2z$ is used to convert Eq. (3) to Eq. (10). Consequently, the constant parameters are as follows

$$\begin{cases} c=p+1, \\ a_i+b_i=p+q+1, \quad i=1,2, \\ a_ib_i=\lambda_i. \end{cases} \tag{11}$$

The hypergeometric function ${}_2F_1(a,b,c;x)$ is given by

$${}_2F_1(a,b,c,x)=1+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}. \tag{12}$$

Function $w(a,b,c;x)$ is as follows [14]

$$w(a,b,c;x)=x^{1-c}\cdot {}_2F_1(a-c+1,b-c+1,2-c;x), \quad \text{if } c>0,\ c \text{ noninteger}, \tag{13}$$

$$w(a,b,c;x)={}_2F_1(a,b,1;x)\ln x+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}x^n\Psi(a,b,n,0), \quad \text{if } |x|<1,\ c=1, \tag{14}$$

$$w(a,b,c;x)={}_2F_1(a,b,c;x)\ln x+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(n!)(c)_n}x^n\Psi(a,b,n,c-1)-\sum_{n=1}^{c-1}\frac{(n-1)!(1-c)_n}{(1-a)_n(1-b)_n}x^{-n}, \quad \text{if } c \text{ natural}, \tag{15}$$

where

$$\Psi(a,b,n,m)=\psi(a+n)-\psi(a)+\psi(b+n)-\psi(b)-\psi(m+n+1)+\psi(m+1)-\psi(n+1)+\psi(1). \tag{16}$$

Function ψ is the logarithmic derivative of Γ function [14]. The Pochhammer symbol or rising factorial $(a)_n$ is given by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)\cdots(a+n-1) \tag{17}$$

3. Eigenvalue singular problem

"The eigenvalue problems associated to spectral type Eq. (1) can be either regular $-1<x_1\leqslant x\leqslant x_2<1$, singular at one end such as $-1<x_1\leqslant x<1$, or singular at both ends $-1<x<1$. In the first two cases, eigenfunctions and eigenvalue equations are written in terms of hypergeometric functions [13]". In what follows, the third case, called hereafter eigenvalue singular problem, is reported. It is showed that in this case, the eigenfunctions are orthogonal polynomials. The eigenvalue singular problem, $-1<x<1$, is given by Eq. (1) and the following boundary conditions

$$y(-1), \quad y(1) \quad \textit{finite}. \tag{18}$$

The eigenfunctions of this problem are showed to be Jacobi orthogonal polynomials, and their corresponding eigenvalues are found. This is in good agreement with Ref. [12] where the same eigenvalues and eigenfunctions have been obtained by other means.

3.1. Singular boundary conditions

Lemma 1. *The solution of* Eq. (1) *for* $p\geqslant 1,\ q\geqslant 1$, *given by* Eq. (9), *is finite at* $x=1$ *if and only if the solution is given by*

$$y(x)=A_1\cdot {}_2F_1\left(a_1,b_1,c;\frac{1-x}{2}\right)+A_2\cdot {}_2F_1\left(a_2,b_2,c;\frac{1-x}{2}\right). \tag{19}$$

Proof. The solution given by Eq. (19) is finite at $x = 1$ and satisfies Eq. (1). □

If the general solution given by Eq. (9) is finite at $x=1$, it is showed by contradiction that it reduces to Eq. (19). Assume B_1 and B_2 are not zero. Then the solution given by Eq. (9) is proved to not be finite at $x=1$, contradicting the assumption. The proof consists of two cases: $p > 1$ and noninteger, and $p = 1,2,3,4\ldots$ In what follows, since the first two terms of Eq. (9) are finite at $x=1$, only the last two terms including $w[a_i,b_i,c;(1-x)/2],\ i=1,2$ functions are to be discussed.

Case 1 (*$p > 1$ and noninteger*). In this case c is a noninteger, Eq. (11), and therefore $w[a_i,b_i,c;(1-x)/2],\ i=1,2$ functions are given by Eq. (13). The limit of the linear combination of terms three and four of Eq. (9) is therefore given by

$$\lim_{\substack{x\to 1\\ a<1}} \frac{\sum_{i=1}^{2} B_i \cdot {}_2F_1\left(a_i - c + 1,\ b_i - c + 1,\ 2 - c; \frac{1-x}{2}\right)}{\left(\frac{1-x}{2}\right)^p}. \tag{20}$$

This limit has to be finite. As the denominator approaches zero since $p > 1$, then a necessary condition is the limit of the numerator to be zero as well. This leads to

$$B_1 + B_2 = 0. \tag{21}$$

Substituting Eq. (21) into Eq. (20), and using l'Hospital rule to find the limit, one obtains

$$\lim_{\substack{x\to 1\\ x<1}} \frac{B_1 \sum_{i=1}^{2} (-1)^{i+1} (a_i - c + 1)(b_i - c + 1) \cdot {}_2F_1\left(a_i - c + 2, b_i - c + 2, 3 - c \frac{1-x}{2}\right)}{p(2-c)\left(\frac{1-x}{2}\right)^{p-1}}. \tag{22}$$

As the denominator approaches zero since $p > 1$, for the limit to be finite, the limit of the numerator has to be zero as well, which leads to $a_1 b_1 = a_2 b_2$ (Eq. (11) is used), or in another form

$$\lambda_1 = \lambda_2, \quad \text{which is false.} \tag{23}$$

Eq. (23) contradicts Eq. (7), λ_1 and λ_2 cannot be equal. Therefore in this case the limit is not finite, which is in contradiction with the condition that Eq. (9) is finite at $x = 1$.

Case 2 ($p = 1, 2, 3, \ldots$). In this case $c = 2, 3, 4, \ldots$, Eq. (11), so the $w[a_i, b_i, c; (1-x)/2]$, $i = 1, 2$ functions are given by Eq. (15). Then, the limit of the linear combination of the not finite terms of terms three and four of Eq. (9) is given by

$$\lim_{\substack{x\to 1\\ x<1}} \frac{\sum_{i=1}^{2} B_i \cdot \left[{}_2F_1\left(a_i, b_i, c; \frac{1-x}{2}\right)\left(\frac{1-x}{2}\right)^p \ln\frac{1-x}{2} - \sum_{n=1}^{p} \frac{(n-1)!(1-c)_n}{(1-a_i)_n (1-b_i)_n} \left(\frac{1-x}{2}\right)^{p-n}\right]}{\left(\frac{1-x}{2}\right)^p}. \tag{24}$$

As the denominator approaches zero, for the limit to be finite, a necessary condition is that the limit of the numerator is zero as well, which leads to

$$\frac{B_1}{(1-a_1)_p (1-b_1)_p} + \frac{B_2}{(1-a_2)_p (1-b_2)_p} = 0. \tag{25}$$

Substitute Eq. (25) into Eq. (24) and cancel a $(1-x)/2$ factor. If $p = 1$ only the logarithm remains in the expression, so its coefficient must be zero. However, for all $p = 1, 2, 3, \ldots$, the limit of the resulting expression is finite only if Eq. (23) is satisfied. This is a contradiction, just as in Case 1.

To conclude, the assumption that B_1 and B_2 are not zero is false. Therefore the solution is given by Eq. (19).

Lemma 2. *If the solution given by* Eq. (19) *is finite at* $x = -1$, *then it must be a polynomial.*

Proof. If the general solution given by Eq. (19) is finite at $x = -1$, then it is proved by contradiction that it reduces to a polynomial. Assume that the solution $y(x)$ given by Eq. (19) is not a polynomial but a series. As x approaches -1, it is showed that the condition of $y(x)$ being finite $x = -1$ leads to a contradiction. Therefore the assumption that $y(x)$ is not a polynomial is false. To prove this, two cases are discussed: $q > 1$ and noninteger, and $q = 1, 2, 3, \ldots$ □

Case 1 (*q > 1 and noninteger*). Using [14; 15.3.3], the limit of the solution given by Eq. (19) becomes

$$\lim_{\substack{x\to -1\\ x>-1}} \frac{\sum_{i=1}^{2} A_i \cdot {}_2F_1\left(c - a_i, c - b_i, c; \frac{1-x}{2}\right)}{\left(\frac{1+x}{2}\right)^q}. \tag{26}$$

Since the limit has to be finite, and the limit of the denominator is zero, a necessary condition is that the limit of the numerator is zero as well, which leads to

$$\sum_{i=1}^{2} A_i \cdot {}_2F_1(c - a_i,\ c - b_i,\ c; 1) = 0. \tag{27}$$

The hypergeometric series in Eq. (27) are convergent, since $q > 1$ and therefore Gauss' theorem [14 15.1.20] is satisfied. Next, using l'Hospital rule to find the limit of Eq. (26), one obtains

$$\lim_{\substack{x\to -1\\ x>-1}} \frac{-\sum_{i=1}^{2} A_i \frac{(c-a_i)(c-b_i)}{c} \cdot {}_2F_1\left(c-a_i+1,\ c-b_i+1,\ c+1\,;\frac{1-x}{2}\right)}{q\left(\frac{1+x}{2}\right)^{q-1}} \tag{28}$$

Since $q > 1$ the denominator approaches zero. For the limit to be finite, the numerator has to approach zero as well, which leads to the following equation

$$\sum_{i=1}^{2} A_i(c-a_i)(c-b_i)\cdot {}_2F_1(c-a_i+1,\ c-b_i+1,\ c+1;1) = 0. \tag{29}$$

The hypergeometric series in Eq. (29) are also convergent [14 15.1.20] since $q > 1$. The system of Eqs. (27) and (29) has a nontrivial solution only if the determinant of the coefficient matrix is zero. After calculations, this leads to Eq. (23) which is false.

Case 2 ($q = 1,2,3,4,\ldots$). From Eq. (11), it results

$$c = a_i + b_i - q. \tag{30}$$

Using [14; 15.3.12 for $m = q$ integer] the limit of the solution given by Eq. (19) can be written as

$$\lim_{\substack{x\to -1\\ x>-1}} \left(\frac{1+x}{2}\right)^{-q} \sum_{i=1}^{2} A_i \cdot \left\{ \frac{\Gamma(q)\Gamma(a_i+b_i-q)}{\Gamma(a_i)\Gamma(b_i)} \sum_{n=0}^{q-1} \frac{(a_i-q)_n(b_i-q)_n}{n!(1-q)_n}\left(\frac{1+x}{2}\right)^{n} - \frac{(-1)^q\Gamma(a_i+b_i-1)}{\Gamma(a_i-1)\Gamma(b_i-1)} \sum_{n=0}^{\infty} \frac{(a_i)_n(b_i)_n}{n!(n+q)!}\left(\frac{1+x}{2}\right)^{n+q} \cdot \left[\ln\frac{1+x}{2} - \psi(n+1) - \psi(n+q+1) + \psi(a_i+n) + \psi(b_i+n)\right]\right\} \tag{31}$$

As the limit has to be finite, and the limit of the denominator is zero, necessarily the limit of the numerator has to be zero as well, which leads to

$$A_1 \frac{\Gamma(q)\Gamma(a_1+b_1-q)}{\Gamma(a_1)\Gamma(b_1)} + A_2 \frac{\Gamma(q)\Gamma(a_2+b_2-q)}{\Gamma(a_2)\Gamma(b_2)} = 0. \tag{32}$$

Use Eq. (11), substitute Eq. (32) into Eq. (31), and cancel a $(1+x)/2$ factor. The limit of the numerator of the resulting expression has to be zero, which after calculations leads to Eq. (23) which is false.

To conclude, the assumption that the solution given by Eq. (19) is not a polynomial is false. Therefore the solution given by Eq. (19) must be a polynomial to ensure convergence for $q \geqslant 1$.

3.2. Eigenvalues and eigenfunctions

Proposition. *Consider the eigenvalue singular problem given by* Eq. (1) *with μ positive and $p \geqslant 1,\ q \geqslant 1$, and*

$$y(-1),\ y(1) \quad \textit{finite}. \tag{33}$$

This eigenvalue singular problem has the following eigenvalues

$$\mu_n = n(n+p+q+1)(n-1)(n+p+q+2), \tag{34}$$

and eigenfunctions

$$y_n(x) = P_n^{(p,q)}(x), \tag{35}$$

where $P_n^{(p,q)}(x)$ are Jacobi orthogonal polynomials [14].

Proof. The general solution of Eq. (1) is given in terms of hypergeometric functions by Eq. (9). The boundary condition $y(1)$ *finite* is satisfied if and only if the general solution reduces to Eq. (19), Lemma 1. The boundary condition $y(-1)$ *finite* is satisfied if and only if Eq. (19) reduces to a polynomial, Lemma 2. The hypergeometric series of Eq. (19) reduce to polynomials if and only if the parameters a_1, a_2 (or b_1, b_2) are negative integers. This way both series ${}_2F_1(a_i, b_i, c_i; 1)\quad i = 1,2$ are finite, see Eq. (17). □

Next, it is showed by contradiction that only one of the parameters a_1, a_2 can be a negative integer. Assume that both a_1, a_2 are negative integers.

$$a_1 = -m, \quad a_2 = -n \quad m, n \text{ natural}. \tag{36}$$

Therefore ${}_2F_1(a_i, b_i, c_i; 1),\quad i = 1,2$, become finite sums, Eq. (17), therefore convergent. Using Eqs. (36) and (11), one obtains

$$b_1 = m + p + q + 1, \quad b_2 = n + p + q + 1, \tag{37}$$

and

$$\lambda_1 = -m(m + p + q + 1), \quad \lambda_2 = -n(n + p + q + 1). \tag{38}$$

Because m and n are natural numbers, and p and q are positive, from Eq. (38) it results that

$$\lambda_1 < 0, \quad \lambda_2 < 0, \tag{39}$$

which contradicts Eq. (7); not both λ_1, λ_2 can be negative. Therefore, the assumption that both a_1, a_2 are negative integers, Eq. (36), is false. Consequently, only one of them can be a negative integer, say a_2. As a_1 is a positive integer, its series ${}_2F_1(a_1, b_1, c_1; 1)$ is divergent. Since $y(-1)$ is finite, the coefficient A_1 of this series in Eq. (19) must be zero

$$A_1 = 0. \tag{40}$$

Therefore, the solution of the eigenvalue singular problem given by Eqs. (1) and (33) is given by

$$y_n(x) = A \cdot {}_2F_1\left(-n, n + p + q + 1, p + 1; \frac{1 - x}{2}\right), \tag{41}$$

as resulting from Eqs. (19) and (40), and $a_2 = -n$. The eigenfunctions $y_n(x)$ given by Eq. (41) are Jacobi polynomials [15], and the corresponding λ_{2n} eigenvalues are given by Eq. (38). Thus the eigenfunctions of the eigenvalue singular problem given by Eqs. (1) and (33) are as follows

$$y_n(x) = A \cdot P_n^{(p,q)}(x) \tag{42}$$

The corresponding eigenvalues result from Eqs. (7), (8), and (38) as

$$\mu_n = n(n + p + q + 1)(n - 1)(n + p + q + 2). \tag{43}$$

This shows that from the general solution expressed in terms of hypergeometric functions, the eigenfunctions of the eigenvalue singular problem result as Jacobi polynomials. The eigenfunctions given by Eq. (42) and the eigenvalues given by Eq. (43) are in agreement with data published in the literature. For the particular case given by Eq. (4), the eigenvalues given by Eq. (49) from Ref. [12] become the eigenvalues given by Eq. (43), and the orthogonal polynomials from Ref. [12] become Jacobi polynomials as in Eq. (42).

4. Application: free bending vibrations of nonuniform beams

The beams considered in this section have circular cross-section, parabolic radius variation with the longitudinal coordinate, and both ends sharp. The boundary conditions have to be free–free since the ends are sharp and consequently cannot sustain any bending moment or shear force. Natural frequencies and mode shapes of transverse vibrations are to be found.

4.1. Equation of transverse vibrations of beams of parabolic radius variation

The dimensionless differential equation of transverse vibrations of Euler–Bernoulli beams is [12]

$$\frac{d^2}{dx^2}\left[I(x)\frac{d^2y(x)}{dx^2}\right] - \frac{\rho_0 \ell^4 \omega^2}{E} A(x) y(x) = 0, \tag{44}$$

where x is the dimensionless longitudinal coordinate of the beam, taken as the dimensional longitudinal coordinate divided by ℓ which is a reference length. The mode shape of vibration is $y(x)$, Young modulus E, cross-section moment of inertia $I(x)$, cross-section area $A(x)$, density ρ_0, and frequency ω. Consider beam of circular cross-section of length 2ℓ whose radius is given by

$$R(x) = R_0(1 - x^2) \quad -1 < x < 1, \tag{45}$$

where R_0 is the reference radius. Thus the area $A(x)$ and the moment of inertia $I(x)$ of the current cross-section are $A(x) = A_0(1 - x^2)^2 \quad I(x) = I_0(1 - x^2)^4$, where A_0 and I_0 are the reference corresponding cross-section quantities $A_0 = \pi R_0^2 \quad I_0 = \pi R_0^4/4$. Substituting into Eq. (44), the following differential equation of motion results as

$$\frac{1}{(1 - x^2)^2}\frac{d^2}{dx^2}\left[(1 - x^2)^4 \frac{d^2y(x)}{dx^2}\right] - \mu \cdot y(x) = 0, \quad \mu = \frac{\rho_0 A_0 \ell^4}{EI_0}\omega^2. \tag{46}$$

4.2. General solution using factorization method [13]

Comparing Eqs. (1) and (46) it results

$$p = 2, \quad q = 2. \tag{47}$$

According to Eqs. (1), (3), (4), (7), and (8), the fourth order differential Eq. (46) can be factorized and its solution can be found solving two second order differential equations given by

$$(1 - x^2)\frac{d^2y(x)}{dx^2} - 4x\frac{dy(x)}{dx} + 2 + (-1)^i\sqrt{4+\mu}y(x) = 0, \quad i = 1, 2. \tag{48}$$

Therefore, the general solution of Eq. (46) is given by Eq. (9) for

$$a_i = \frac{5+\sqrt{25-4\lambda_i}}{2}, b_i = \frac{5-\sqrt{25-4\lambda_i}}{2} \quad i = 1, 2 \tag{49}$$

where λ_i are found from Eqs. (7), (8) and (47).

4.3. Natural frequencies and mode shapes of beam sharp at both ends

The free-free boundary conditions of the beam are the conditions of zero bending moment M and zero shear force T at the ends as follows

$$M(-1) = M(1) = T(-1) = T(1) = 0. \tag{50}$$

Caruntu [10] showed that for beams sharp at both ends the boundary conditions (42) reduce to conditions of finite displacements of the free sharp ends Eq. (33). Consequently, the boundary value problem given by Eqs. (46) and (33) is an eigenvalue singular problem given by the proposition. According to Eqs. (34) and (35) of the proposition, the eigenfunctions $y_n(x)$ and eigenvalues μ_n are given by

$$y_n(x) = P_n^{(2,2)}(x), \tag{51}$$

$$\mu_n = n(n-1)(n+5)(n+6). \tag{52}$$

Consequently, from Eqs. (51) and (52) where n has been shifted to $(n+1)$ at the right hand sides, the eigenfrequencies (natural frequencies ω_n) and the mode shapes $Y_n(x)$ of the beam are given by

$$Y_n(x) = P_{n+1}^{(2,2)}(x), \tag{53}$$

$$\omega_n = \sqrt{n(n+1)(n+6)(n+7)}\sqrt{\frac{EI_0}{\rho_0 A_0}}. \tag{54}$$

This shifting allows for the natural frequencies and the mode shapes of the beam to be given starting from $n = 1$.

5. Discussion and conclusion

In this paper, the perfect agreement between Refs. [12,13] is proved. It is showed that the hypergeometric series solutions reported in [13] reduce to classical Jacobi orthogonal polynomials reported in Ref. [12] for the associated eigenvalue singular problems. Moreover, a mechanical engineering example of transverse vibration of nonuniform beams (1) illustrates how dynamic modal characteristics (modes of vibration) for beams sharp at both ends can be found from its general solution obtained by factorization method, and (2) shows the same results as Ref. [12].

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